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A. M. Mathai and H. J. Haubold
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The entropic, distributional and differential pathways to model building

A. M. Mathai[1] & **H. J. Haubold**[2]

[1] **Director,**
Centre for Mathematical Sciences,India,
Arunapuram P.O., Palai, Kerala 686 574, India
and Emeritus Professor, McGill University, Montreal, Canada.
(directorcms458@gmail.com; cmspala@gmail.com; mathai@math.mcgill.ca)
and
[2] **Office of Outer Space Affairs, United Nations,**
P.O. Box 500, Vienna International Centre, A-1400 Vienna, Austria.
(hans.haubold@unvienna.org;hans.haubold@unoosa.org)

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Objectives

- The pathway idea is a way of going from one family of functions to another family of functions and yet another family of functions through a parameter in the model so that a switching mechanism is introduced into the model through a parameter.
- The advantage of the idea is that the model can cover the ideal or stable situation in a physical situation as well as cover the unstable neighborhoods or move from unstable neighborhoods to the stable situation.
- The basic idea is illustrated for the real scalar case here and its connections to the hot topics in astrophysics and non-extensive statistical mechanics namely superstatistics and Tsallis statistics, Mittag-Leffler models, hypergeometric fucntions and generalized special functions such as H-function etc are pointed out.
- At each generalization, its connections to various quantities in different disciplines are pointed out.
- Pathway idea is available for the real and complex rectangular matrix variate cases but only the real scalar case is illustrated here.

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- From the 1980's applications of all these techniques into astrophysics problems were explored in the areas of energy generation, solar and stellar models, gravitational instability problem, neutrino problem and so on, [Books: Mathai and Haubold (1988), Mathai (1993), Mathai and Haubold (2008), Mathai, Saxena and Haubold (2010)].
- A lot of interesting results, mathematically and statistically and perhaps with potential of physical interpretations, could be obtained by the fusion of special function theory, statistical distribution theory, information theory, characterizations and astrophysics.
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- Consider a general input-output type situation.
- It could be reactions producing new particles, diffusion or destruction of some particles and thus the residual part is what is observed, it could be an industrial production unit where input may be the money value of the raw materials put in and the output may be the money value of the final product and so on.
- Consider particle reactions and let $N(t)$ be the number density at time t and the rate of reaction denoted by $\frac{\mathrm{d}N(t)}{\mathrm{d}t}$.
- If the number of particles produced is proportional to the original population size then the differential equation is $\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \lambda\, N(t)$ where λ denotes the rate of reactions.
- Let the diffusion rate or destruction rate be μ then the residual rate is $c = \lambda - \mu$. If production dominates then $c > 0$ and if destruction dominates then $c < 0$.
- Then for the model

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -c\, N(t) \Rightarrow N(t) = N_0\, \mathrm{e}^{-ct} \tag{1.1}$$

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- If the rate of change is proportional to a power of the population size and if decay dominates then the equation and the solution are the following:

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = -c[N(t)]^{\alpha} \Rightarrow N(t) = -[1-c(1-\alpha)t]^{\frac{1}{1-\alpha}}. \quad (1.2)$$

This is a power law type of behavior.

- For $\alpha < 1$ the function in (1.2) belongs to a particular case of a type-1 beta family of functions. Let $N(t)$ in (1.2) be denoted by $N_1(t)$.
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- Here (1.3) is a special case of a type-2 beta family of functions.
- When $\alpha \to 1$, denoting $N(t)$ by $N_3(t)$ in this case,

$$N_3(t) = \lim_{t\to 1_+} N_2(t) = \lim_{t\to 1_-} N_1(t) = \mathrm{e}^{-ct}. \quad (1.4)$$

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- $N_1(t)$ for $\alpha < 1$ and $N_2(t)$ for $\alpha > 1$ describe a wide range of models.
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Optimization of Entropy

- Models in physical situations are also constructed by optimizing entropy measures.
- The Shannon entropy in a probability scheme, for the continuous situation is

$$S(f) = -k \int_{-\infty}^{\infty} f(x) \ln f(x) \mathrm{d}x \qquad (2.1)$$

where $f(x)$ is a statistical density and k is a constant.
- When k is present, we can assume $f(x)$ to be any non-negative integrable function. S represents a *measure of uncertainty* in a probability scheme.
- If $S(t)$ is maximized over all functional f satisfying the condition $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$ and $f(x) \geq 0$ for all x then f is the uniform density.
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Optimization of Entropy

- Models in physical situations are also constructed by optimizing entropy measures.
- The Shannon entropy in a probability scheme, for the continuous situation is

$$S(f) = -k\int_{-\infty}^{\infty} f(x)\ln f(x)\mathrm{d}x \tag{2.1}$$

where $f(x)$ is a statistical density and k is a constant.

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$$M_\alpha(f) = \frac{[\int_{-\infty}^{\infty}(f(x))^{2-\alpha}\mathrm{d}x - 1]}{\alpha - 1}, \alpha \neq 1, \alpha \leq 2. \quad (3.2)$$

- Consider the optimization of (3.2) subject to the conditions

$$(a): \int_{-\infty}^{\infty} |x|^{\delta} f(x)\mathrm{d}x = k_1 < \infty,$$

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where k_1 and k_2 are fixed, and the optimization is done over all non-negative integrable functions.
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$$g(f) = [f(x)]^{2-\alpha} - \lambda_1 |x|^{\gamma} f(x) + \lambda_2 |x|^{\gamma+\delta} f(x)$$

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- Note that (3.3) for $\alpha < 1, a > 0, \delta > 0, x > 0$ can be called an extended generalized type-1 beta model.
- For $\alpha > 1$, writing $1 - \alpha = -(\alpha - 1)$, (3.3) reduces to the following:

$$f_2(x) = c_2|x|^{\gamma}[1 + a(\alpha - 1)|x|^{\delta}]^{-\frac{1}{\alpha-1}}, \alpha > 1, \delta > 0, a > 0. \tag{3.4}$$

- Note that (3.4) can be called an extended generalized type-2 beta model.
- Denoting $f(x)$ under $\alpha < 1$ as $f_1(x)$ we have

$$f_3(x) = \lim_{\alpha \to 1_-} f_1(x) = \lim_{\alpha \to 1_+} f_2(x) = c_3|x|^{\gamma} e^{-a|x|^{\delta}} \tag{3.5}$$

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$$c_1 = \frac{[a(1-\alpha)]^{\frac{\gamma+1}{\delta}}}{2} \frac{\Gamma(\frac{\gamma+1}{\delta} + \frac{1}{1-\alpha} + 1)}{\Gamma(\frac{\gamma+1}{\delta})\Gamma(\frac{1}{1-\alpha} + 1)}, \alpha < 1, a > 0, \delta > 0, \gamma + 1 > 0 \quad (3.6)$$

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- The model in (3.3) for a general α is the scalar version of the pathway model of Mathai (2005).This is the distributional pathway.
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- When $\alpha < 1$ then the model describes the whole family of functions belonging to extended generalized type-1 beta family.
- When $\alpha > 1$ then we move into the whole family of functions belonging to the extended generalized type-2 beta family.
- When $\alpha \to 1$ then both these families go into the family of extended generalized gamma family.
- This is the differential pathway.
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- The model in (3.5) for a prefixed parameter a can be written as a conditional density of the type

$$f_4(x|a) = \frac{a^{\frac{\gamma+1}{\delta}}}{2\Gamma(\frac{\gamma+1}{\delta})}|x|^{\gamma}\mathrm{e}^{-a|x|^{\delta}}, a>0, -\infty<x<\infty. \tag{4.1}$$

- Suppose that the parameter a has a prior density given by

$$g(a) = \frac{1}{\eta^{\epsilon}\Gamma(\epsilon)}a^{\epsilon-1}\mathrm{e}^{-\frac{a}{\eta}}, a>0, \eta>0, \epsilon>0 \tag{4.2}$$

where ϵ and η are known constants.

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Fractional Considerations

- Going back to our basic growth-decay problem where the rate of change is proportional to the population size, our basic differential equation, equation (1.1), is
$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) = -c\,f(t), c > 0 \Rightarrow f(t) - f_0 = -c\int f(t)\mathrm{d}t. \quad (5.1)$$

- If the total integral is replaced by a fractional integral of the Riemann-Liouville type let us see what happens. The left sided Riemann-Liouville fractional integral operator is denoted by ${}_0D_x^{-\alpha} = {}_0I_x^{\alpha}$ and it is defined as
$${}_0D_x^{-\alpha}f = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\mathrm{d}t, \Re(\alpha) > 0. \quad (5.2)$$

- Fractional integral can be given many interpretations in statistical literature as fraction of a total integral, as the density of residual variable $u = x - y$ where x and y are independently distributed real positive random variables such that $x - y > 0$ etc [Mathai (2010), Seema Nair (2010)].
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Fractional Considerations

- Let the Laplace parameter be s. Let the Laplace transform of f be denoted by $\tilde{f}(s)$.
- Then

$$L_f(s) - f_0 \int_0^\infty e^{-sx} dx = -c \int_{x=0}^\infty e^{-sx} [\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt] dx.$$

Then

$$\tilde{f} - \frac{f_0}{s} = -s^{-\alpha} \tilde{f}(x) \Rightarrow \tilde{f} = \frac{f_0}{s[1 + cs^{-\alpha}]} \tag{5.4}$$

$$= f_0 \sum_{k=0}^\infty (\frac{c}{s^\alpha})^k (-1)^k.$$

Taking the inverse Laplace transform we have

$$f(x) = f_0 \sum_{k=0}^\infty (-1)^k \frac{c^k x^{\alpha k}}{\Gamma(1 + \alpha k)} = f_0 E_\alpha(-cx^\alpha) \tag{5.5}$$

where $E_\alpha(\cdot)$ is the basic Mittag-Leffler function.

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- Generalization of the basic Mittag-Leffler function are the following:

$$E_{\alpha}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(1+\alpha k)}, \Re(\alpha)>0,\; E_1(x)=e^x$$

$$E_{\alpha,\beta}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\beta+\alpha k)}, \Re(\alpha)>0, \Re(\beta)>0$$

$$E_{\alpha,\beta}^{\gamma}(x)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!}\frac{x^k}{\Gamma(\beta+\alpha k)}, \Re(\alpha)>0, \Re(\beta)>0 \qquad (5.6)$$

where $(\gamma)_k$ is the Pochhammer symbol

$$(\gamma)_k=\gamma(\gamma+1)...(\gamma+k-1), (\gamma)_0=1, \gamma\neq 0.$$

- More generalized form of (5.5) is the Wright's function, which is a special case of the H-function.
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- It is seen that when we move from a total differential equation to a fractional differential equation, Mittag-Leffler function and its generalizations, Wright function and H-function enter into the solutions.
- A series of recent papers are available on the solutions of fractional reaction equations and fractional reaction-diffusion equations.
- The Laplace transform in (5.4) belongs to a general class of Laplace transforms, see Mathai et al. (2006) and the various references therein, and various members from this general class appear when solving some fractional differential equations.
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Fractional Considerations

- Let us see what happens if a parameter is becoming larger and larger in a Mittag-Leffler model of (5.6).
- Suppose that β is real and it is becoming larger and larger.
- Then by using the asymptotic expansion of gamma functions or as a first approximation the Stirling's formula

$$\Gamma(z+a) \approx \sqrt{2\pi} z^{z+a-\frac{1}{2}} e^{-z} \text{ for } |z| \to \infty, \ a \text{ is bounded}$$

- we see that

$$\Gamma(\beta) E^{\gamma}_{\delta,\beta}(a(\beta x)^{\delta}) \approx \sum_{k=0}^{\infty} \frac{a^k (\gamma)_k x^{\delta k}}{k!} \frac{\sqrt{2\pi}\beta^{\beta-\frac{1}{2}} e^{-\beta}}{\sqrt{2\pi}\beta^{\beta-\frac{1}{2}+\delta k} e^{-\beta}}$$

$$= \sum_{k=0}^{\infty} \frac{a^k (\gamma)_k}{k!} ((\frac{x}{\beta})^{\delta})^k = (1+ax^{\delta})^{-\gamma}. \tag{5.7}$$

- This is the pathway model, Tsallis statistics and superstatistics for the case $\alpha > 1$ for $\gamma = \frac{1}{\alpha-1}, a = b(\alpha-1), b > 0, \alpha > 1, \delta > 0$. For $\gamma = 1$, (5.7) becomes a power series.

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A Mathematical Perspective

- Mathematically speaking the whole process of transition from one functional form to another, Tsallis statistics, superstatistics and pathway models in the scalar case can be described as getting rid off some parameters from a hypergeometric series.
- Take for example a ${}_1F_1$ series:

$$ {}_1F_1(a;b;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\frac{x^k}{k!}. \tag{6.1} $$

- If we wish to get rid off an upper or lower parameter then we do a limiting process.
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$$ \lim_{a\to\infty} {}_1F_1(a;b;\frac{x}{a}) = {}_0F_1(\,;b;x) $$

$$ \lim_{b\to\infty} {}_1F_1(a;b;bx) = {}_1F_0(a;\,;x), \ |x|<1 $$

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$$\lim_{a\to\infty}\frac{(a)_k}{a^k}=1=\lim_{a\to\infty}\frac{a^k}{(a)_k}. \qquad (6.4)$$

- All these ideas are extended to the matrix-variate cases, to real positive definite, hermitian positive definite and to rectangular matrices, see the basic paper Mathai (2005), and later papers by the author and his co-workers are also available.
- One such model is the following:

$$f(X)=c\,|A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{\gamma}|I-a(1-\alpha)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{\frac{\eta}{1-\alpha}} \qquad (6.4)$$

- where X is a $p\times r,\ r\geq p$ matrix of full rank p of distinct real random or mathematical variables, A is a $p\times p$ constant positive definite matrix, B is a $r\times r$ constant positive definite matrix, X' denotes the transpose of X, $A^{\frac{1}{2}}$ denotes the positive definite square root of the positive definite matrix A, $f(X)$ is a real-valued scalar function of X and c is a constant.
- This c can act as a normalizing constant if $f(X)$ is treated as a statistical density.
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- If the matrix X is relocated at some other matrix M then replace X by $X - M$ in the model.

A Mathematical Perspective

- All the above limiting forms are available by using the fact that

$$\lim_{a\to\infty}\frac{(a)_k}{a^k}=1=\lim_{a\to\infty}\frac{a^k}{(a)_k}. \tag{6.4}$$

- All these ideas are extended to the matrix-variate cases, to real positive definite, hermitian positive definite and to rectangular matrices, see the basic paper Mathai (2005), and later papers by the author and his co-workers are also available.
- One such model is the following:

$$f(X)=c\,|A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{\gamma}|I-a(1-\alpha)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{\frac{\eta}{1-\alpha}} \tag{6.4}$$

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- The constants $\eta > 0,\ a > 0$ and α are real scalars where α is the pathway parameter.
- For $\alpha < 1$ the model in (6.4) will stay in the generalized real matrix-variate type-1 beta family of functions.
- For $\alpha > 1$ the model in (6.4) will go to the generalized real matrix-variate type-2 beta family of functions.
- When $\alpha \to 1$ both these type-1 beta and type-2 beta families will go to a generalized matrix-variate gamma family of functions.
- This can be seen by using the result

$$\lim_{\alpha \to 1} |I - a(1-\alpha)A^{\frac{1}{2}}XBX'A^{\frac{1}{2}}|^{\frac{\eta}{1-\alpha}} = \exp\{-a\eta\,\mathrm{tr}(A^{\frac{1}{2}}XBX'A^{\frac{1}{2}})\}$$

  where $\mathrm{tr}(\cdot)$ denotes the trace of $(\cdot)$.
- It can be seen that all the real matrix-variate densities that are used in the current literature are available from the model (6.4) for various values of the pathway parameter α.
- A similar rich family is there if we consider the transition from a Bessel form to the exponential form. Model, corresponding to the one in (6.4), is available when the variables are in the complex domain also.

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