Basic Space Physics through Problem Solving

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(1) [**Sun**] (Ref. 1, adapted) Estimate the mass lost by the Sun per unit time through electromagnetic radiation, by calculating the value of mass lost in association with on-going fusion reaction to convert hydrogen to helium. In the meantime, calculate the mass lost by the Sun per unit time through the solar wind. If those mass loss rates were constant in time, how long would it take to lose all the solar mass?

(hint)
solar constant: $I = 1.37$ kW/m^2
1 AU = $1.50 \cdot 10^{11}$ m
speed of light: $3.00 \cdot 10^8$ m/s
solar wind speed: $u = 400$ km/s
solar wind density: $n = 7 \cdot 10^6$ m^{-3}
proton mass: $m_p = 1.67 \cdot 10^{-27}$ kg
solar mass: $M_\odot = 1.99 \cdot 10^{30}$ kg

(2) [**Solar wind**] (Ref. 2) A comet's ion tail is composed of cometary ions embedded in the solar wind. Consider an ion tail that points 4° away from the radial direction. What is the solar wind speed if the azimuthal component of the comet's velocity is 30 km/s?

(3) [**Archimedean spiral**] (Ref. 2, 3) Assume that the solar wind speed is $u = 400$ km/s.

1. Find the heliocentric distance where the average interplanetary magnetic field has wrapped itself around once. Give the distance in units of AU.
2. What is the angle between the radius vector and the magnetic field vector at this point?
3. What is the magnitude of the magnetic field at this point if the source surface of the magnetic field is located at $R_S = 10\ R_\odot$ ($R_\odot = 6.96 \cdot 10^8$ m) and if the magnetic field magnitude at the source surface is $B_S = 10^{-6}$ T?
4. Determine the number of times the magnetic field has wound around the Sun by a heliocentric distance of 100 AU.

(hint)

$$\mathbf{B} \approx B_s \left(\frac{R_s}{r}\right)^2 \mathbf{e}_r - B_s \left(\frac{R_s}{r}\right) \frac{\Omega}{u} \mathbf{e}_\varphi \qquad \text{for } r \gg R_s$$

$\Omega = 2.80 \cdot 10^{-6}$ rad/s

1 AU = $1.50 \cdot 10^{11}$ m

(4) [**Interplanetary shock**] (Ref. 4) The passage of a coronal mass ejection (CME) and an associated interplanetary shock preceding it was measured by the HELIOS spacecraft as shown in the following figure.

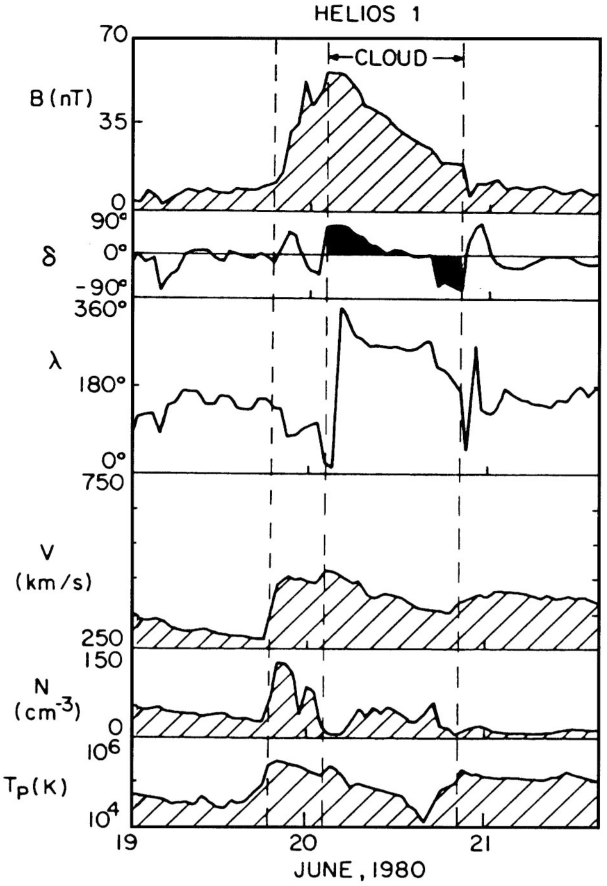

Figure. Data from *HELIOS 1* spacecraft for passage of a magnetic cloud and the interplanetary shock preceding it. The shock is located between the dashed lines near the 20 June mark. (Top panel) Magnetic field strength. (Middle panels) δ and λ: latitude and longitude of magnetic field in solar-ecliptic coordinates. (Lower panels) The measured solar wind speed V, density N, and proton temperature T_p. (From Burlaga et al., 1982.)

The shock jump calculated from the plasma density is about $Z = 3$. From this value of Z and the data shown in the figure, show that the shock speed is $u_s \approx 555$ km/s. Next use this speed to determine the upstream and downstream plasma speeds. Are these speeds consistent with the shock jump Z found from the density data? Show that the upstream Mach number is $M \approx 3$, and also determine the downstream Mach number and the pressure jump across the shock. The Rankine-Hugoniot relations in neutral gas dynamics are adequate enough in this case.

(hint)

$$\frac{u_2}{u_1} = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{(\gamma + 1)M_1^2}$$

$$Z \equiv \frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{\gamma + 1}{\gamma - 1 + \frac{2}{M_1^2}}$$

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1}$$

(1: upstream 2: downstream)

(5) [**Space weather**] (Ref. 1) Assuming that the solar wind flows from the Sun to the Earth at a constant speed, calculate the angle the Sun revolves, as observed from the Earth, during the period of time it takes for the solar wind to reach the Earth. Calculate two cases for the solar wind speeds of 750 km/s and 300 km/s. The Sun's rotation period is 25.4 days.

(hint)
Earth's revolution period = 365.256 days
1 AU = $1.50 \cdot 10^{11}$ m

(6) [**Space weather**] (Ref. 1) Suppose that a solar flare generates X-rays and 10 MeV protons. Calculate the time required for these radiative events to reach the Earth. Assume that the protons take a straight path to the Earth. Also calculate the travel time of the solar wind from the Sun to the Earth at a mean speed of 450 km/s.

(hint)
1 AU = $1.50 \cdot 10^{11}$ m
speed of light: c = $3.00 \cdot 10^{8}$ m/s
proton mass: $m_p = 1.67 \cdot 10^{-27}$ kg
proton charge: e = $1.60 \cdot 10^{-19}$ C

(7) [**Geomagnetic coordinates**] (Ref. 1) The geographic coordinates (λ, φ) of any point on the Earth can be converted to its corresponding geomagnetic coordinates (Λ, Φ) by the following equations.

$$\sin \Lambda = \sin \lambda \sin \lambda_0 + \cos \lambda \cos \lambda_0 \cos(\varphi - \varphi_0)$$

$$\sin \Phi = \frac{\cos \lambda \sin(\varphi - \varphi_0)}{\cos \Lambda}$$

where (λ_0, φ_0) is the geographic coordinates of the north geomagnetic pole.

1. Show these relations using formulae of spherical triangles.
2. Calculate the magnetic latitude of your hometown. According to IGRF 2010, λ_0 = 80.0°, and φ_0= −72.2°.

(hint)

sine law

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

cosine law

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

(8) [**Magnetosphere**] (Ref. 1, 2, 3, adapted) The dipole magnetic field is expressed as

$$B_r = -2B_0 \left(\frac{a}{r}\right)^3 \sin \lambda$$

$$B_\lambda = B_0 \left(\frac{a}{r}\right)^3 \cos \lambda$$

where a is the radius of the Earth, and λ is the magnetic latitude (not the geographic latitude in Problem 7). B_0 is the magnetic field strength at the equator on the Earth.

1. Calculate the total force of the magnetic field.
2. Field lines are expressed by the relation

$$\frac{dr}{rd\lambda} = \frac{B_r}{B_\lambda}$$

Show that the equation for a magnetic field line is given by

$$\frac{r}{a} = L \cos^2 \lambda$$

The dimensionless parameter L is a constant of the field line and represents the geocentric distance (measured by a, the radius of the Earth) where the field line crosses the magnetic equator.

3. A magnetic field line crosses the magnetic equator at 4 R_E. Assuming that the Earth's field is a dipole, where does this magnetic field line intersect the surface of the Earth?
4. Calculate the L value of your hometown.

(9) [**Magnetosphere**] At ionospheric altitudes, the Earth's magnetic field is approximated by a dipole field. Assume that the polar cap boundary is a circle centered on the pole. When the polar cap boundary is located at a latitude of Λ on the surface of the Earth, show that the magnetic flux

that penetrates the polar cap (for one hemisphere) is given by

$$\Phi_{pc} = 2\pi a^2 B_0 \cos^2 \Lambda$$

where a is the radius of the Earth, and B_0 is the magnetic field strength at the equator on the Earth. The polar cap expands during the growth phase of substorms (the auroral oval moves into lower latitudes). If we assume that the radius of the polar cap expands from 15° to 20°, how much does the tail magnetic flux increase?

(10) [**Dessler-Parker-Sckopke relation**] Assuming that the ring current is carried by particles trapped in the dipole field, the magnetic field deviation $\Delta\mathbf{B}$ at the Earth's center caused by the ring current is given by

$$\frac{\Delta\mathbf{B}}{B_0} = -\frac{2}{3}\frac{W_{RC}}{W_{dip}}\mathbf{e}_z$$

where W_{RC} is the total energy of all ring current particles, W_{dip} is the total magnetic energy of the Earth's dipole field outside the surface of the Earth, and $\mathbf{e}_z$ represents the unit vector antiparallel to the dipole vector. Thus, in the zeroth order, D_{st} is proportional to W_{RC}.

1. Show that

$$W_{dip} = \frac{4\pi a^3}{3}\frac{B_0^2}{\mu_0}$$

and calculate the numerical value of W_{dip}.

(hint)
$B_0 = 3.00 \cdot 10^{-5}$ T
$a = 6.38 \cdot 10^6$ m

2. Derive the Dessler-Parker-Sckopke relation assuming a single ion species drifting in the dipole field in the equatorial plane.

(hint)
The gradient B drift at the equator at geocentric distance La is given by

$$\mathbf{v}_B = -\frac{3L^2 w}{qB_0 a}\boldsymbol{e}_\varphi$$

where $w = \frac{1}{2}mv^2$ is the particle energy, and q is the charge of the particle. Also note that each gyrating particle has a magnetic moment μ given by $\mu = \frac{w}{B}$, which also contributes to $\Delta\mathbf{B}$.

References

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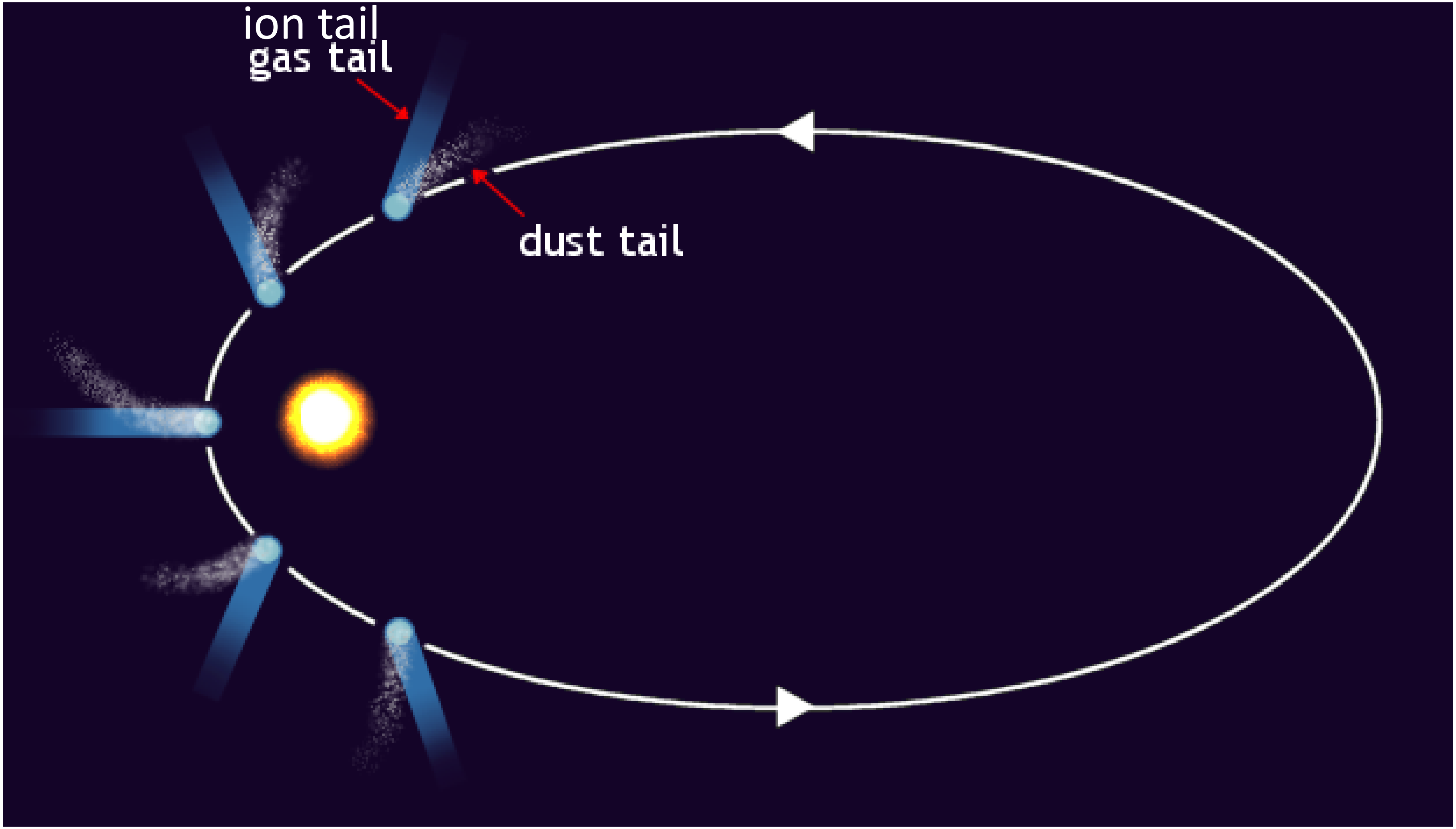
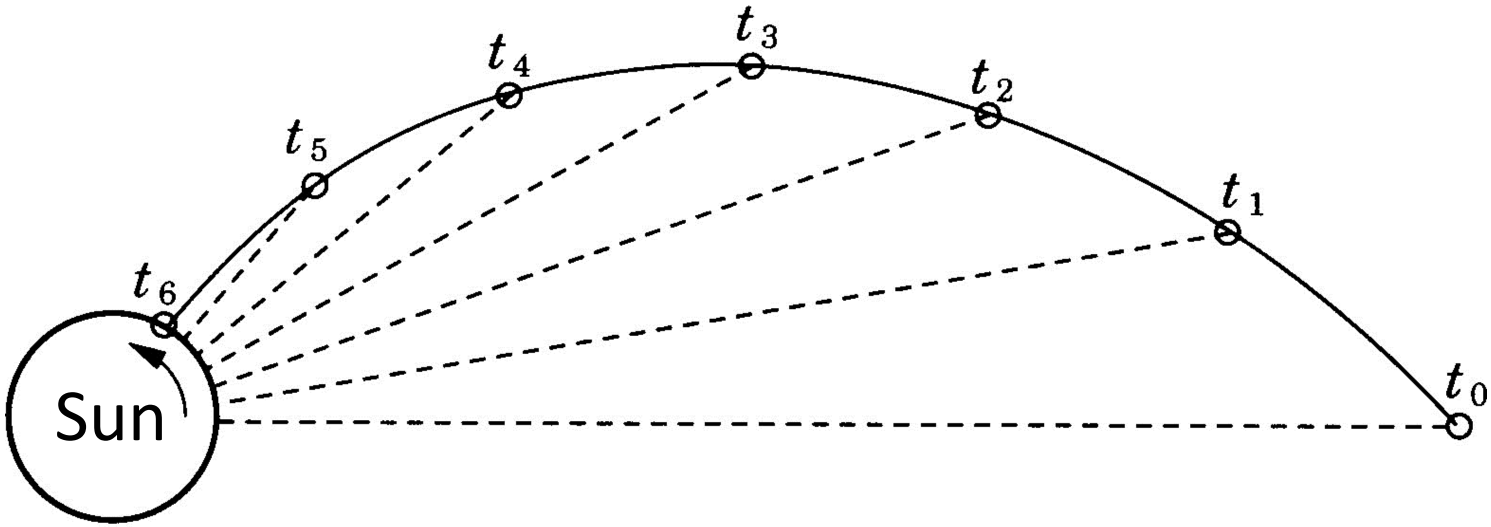
2. Gombosi, T. I., Physics of the Space Environment, Cambridge University Press, 1998.
3. Prölss, G. W., Physics of the Earth’s Space Environment, Springer-Verlag, 2004.
4. Cravens, T. E., Physics of Solar System Plasmas, Cambridge University Press, 1997.

Comet Hale–Bopp (1997)

ion tail
(gas tail)

dust tail


ion tail
gas tail
dust tail



In a frame rotating with the Sun

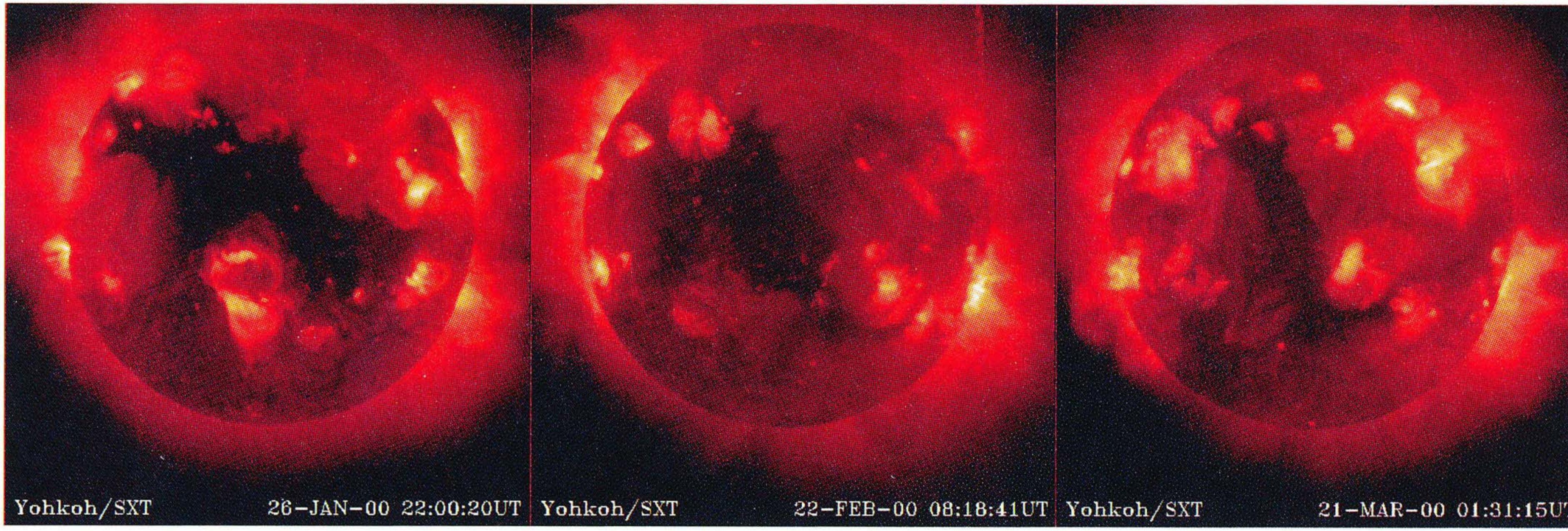
$$r = r_0 + ut \qquad \varphi = \varphi_0 - \Omega t$$

Eliminating t, we obtain $$r = r_0 - \frac{u}{\Omega}(\varphi - \varphi_0)$$

$$\nabla \cdot \mathbf{B} = 0 \rightarrow \frac{1}{r^2}\frac{d}{dr}(r^2 B_r) = 0 \rightarrow B_r \sim \frac{1}{r^2}$$

$$\frac{B_\varphi}{B_r} = \frac{r d\varphi}{dr} \approx -\frac{r\Omega}{u} \rightarrow B_\varphi \sim r B_r \sim \frac{1}{r}$$

Recursive (27-day period) coronal holes



26 January 2000 (day 26 of the year)

22 February 2000 (day 53 of the year)

21 March 2000 (day 81 of the year)

→ CIR-induced geomagnetic storms

(1)

energy loss rate $4\pi R^2 \cdot I = 4 \times 3.14 \times (1.50 \times 10^{11})^2 \times 1.37 \times 10^3$ W

$= 3.87 \times 10^{26}$ W

mass loss rate

$$\dot{m} = \frac{1}{c^2}\dot{E} = \frac{3.87 \times 10^{26}\ \mathrm{W}}{(3.00 \times 10^8\ \mathrm{m/s})^2} = 4.30 \times 10^9\ \mathrm{kg/s}$$

mass loss rate by the solar wind

$$\rho = m_p n = 1.67 \times 10^{-27} \times 7 \times 10^6 = 1.17 \times 10^{-20}\ \mathrm{kg/m^3}$$

$$4\pi R^2 \cdot \rho u = 4 \times 3.14 \times (1.5 \times 10^{11})^2 \times 1.17 \times 10^{-20} \times 4 \times 10^5$$

$$= 1.32 \times 10^9\ \mathrm{kg/s}$$

total mass loss rate

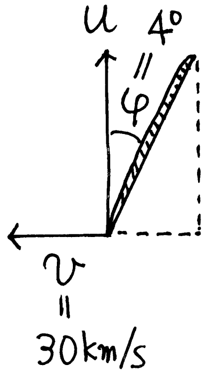
$$(4.30 + 1.32) \times 10^9 = 5.62 \times 10^9\ \mathrm{kg/s}$$

time to lose all mass

$$\frac{1.99 \times 10^{30}\ \mathrm{kg}}{5.62 \times 10^9\ \mathrm{kg/s}} = 3.54 \times 10^{20}\ \mathrm{s} = \frac{3.54 \times 10^{20}}{86400 \times 365.25}\ \mathrm{y}$$

$$= 1.12 \times 10^{13}\ \mathrm{y} \sim 10 \text{ trillion years}$$

(2)



$$u = v \cot\varphi = 30 \cot 4° = 429\ \mathrm{km/s}$$

(3)

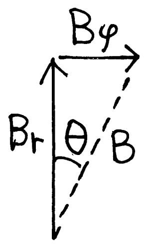
1.
$$\begin{cases} \varphi(r) = \varphi_0 - \dfrac{\Omega}{u}(r - R_s) \\ r = R_s - \dfrac{u}{\Omega}(\varphi - \varphi_0) \approx -\dfrac{u}{\Omega}(\varphi - \varphi_0) \quad \text{for } r \gg R_s \end{cases}$$

Substitute $\varphi = \varphi_0 - 2\pi$ in r

$$r = R_s + \frac{u}{\Omega} \cdot 2\pi \approx \frac{u}{\Omega} \cdot 2\pi$$

$$= \frac{4 \times 10^5 \times 2 \times 3.14}{2.80 \times 10^{-6}} = 8.97 \times 10^{11} \text{ m} = \frac{8.97 \times 10^{11}}{1.50 \times 10^{11}} \text{ AU} = 5.98 \text{ AU}$$

2.



$$\tan\theta = \frac{|B_\varphi|}{B_r} = \frac{\Omega}{u} r$$

$$= \frac{2.80 \times 10^{-6} \times 8.97 \times 10^{11}}{4 \times 10^5} = 6.279$$

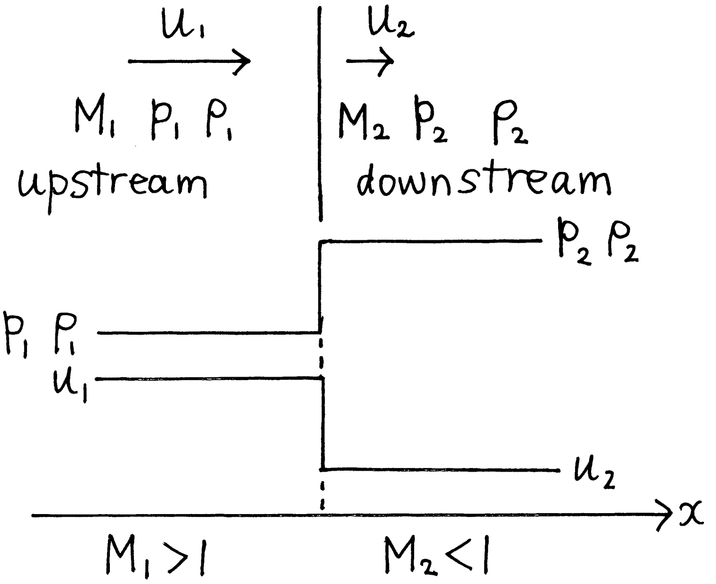
$$\theta = 81.0^\circ$$

3.
$$B = B_s \left(\frac{R_s}{r}\right)^2 \sec\theta = 10^{-6} \left(\frac{10 \times 6.96 \times 10^8}{8.97 \times 10^{11}}\right)^2 \sec 81.0^\circ \text{ T}$$

$$= 3.85 \times 10^{-10} \text{ T} = 0.385 \text{ nT}$$

4.
$$\frac{100}{5.98} = 16.7 \text{ times}$$

(4) Rankine-Hugoniot relations



$$\frac{u_2}{u_1} = \frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M_1^2} \quad ①$$

$$Z \equiv \frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{\gamma+1}{\gamma-1+\frac{2}{M_1^2}} \quad ②$$

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1} \quad ③$$

$\gamma = \frac{5}{3}$ specific heat ratio (monoatomic gas)

Sonic Mach number $\quad M^2 \equiv \frac{\rho u^2}{\gamma P}$

$$Z = \frac{\rho_2}{\rho_1} = \frac{u_1'}{u_2'} = \frac{u_1 - u_s}{u_2 - u_s}$$

From the figure

$Z = 3$, $u_2 = 460$ km/s, $u_1 = 270$ km/s

$$u_s = \frac{Z u_2 - u_1}{Z - 1} = \frac{3 \times 460 - 270}{3 - 1} = 555 \text{ km/s}$$

$$\begin{cases} u_1' = u_1 - u_s = 270 - 555 = -285 \text{ km/s} \\ u_2' = u_2 - u_s = 460 - 555 = -95 \text{ km/s} \end{cases}$$

$$\frac{u_1'}{u_2'} = \frac{285}{95} = 3 \quad \text{consistent with} \quad Z = \frac{\rho_2}{\rho_1} = 3$$

From ②

$$M_1^2 = \frac{2Z}{(\gamma+1) - Z(\gamma-1)} = \frac{2 \times 3}{(\frac{5}{3}+1) - 3(\frac{5}{3}-1)} = 9 \quad \therefore M_1 = 3$$

$$\frac{M_2^2}{M_1^2} = \left(\frac{\rho_2}{\rho_1}\right)\left(\frac{P_1}{P_2}\right)\left(\frac{u_2}{u_1}\right)^2 = Z \cdot \frac{\gamma+1}{2\gamma M_1^2 - (\gamma-1)} \cdot \frac{1}{Z^2}$$

$\uparrow$ ② $\quad \uparrow$ ③ $\quad \uparrow$ ②

$$M_2^2 = \frac{\gamma+1}{2\gamma M_1^2 - (\gamma-1)} \frac{M_1^2}{Z}$$

$$= \frac{\frac{5}{3}+1}{2\times\frac{5}{3}\times 9 - (\frac{5}{3}-1)} \cdot \frac{9}{3} = \frac{3}{11} \qquad \therefore M_2 = \sqrt{\frac{3}{11}} = 0.522$$

From ③

$$\frac{P_2}{P_1} = \frac{2\times\frac{5}{3}\times 9 - (\frac{5}{3}-1)}{\frac{5}{3}+1} = 11$$

(5) The number of times the Sun rotates in a year is

$$\frac{365.256}{25.4}.$$

During this period, the Earth revolves around the Sun once.

The rotational velocity of the Sun as observed from the Earth is

$$\frac{360°\times\left(\frac{365.256}{25.4}-1\right)}{365.256 \text{ days}} = 13.2 \quad °/\text{day}$$

$u = 750$ km/s case

$$13.2\times\frac{1.50\times10^8 \text{ km}}{750 \text{ km/s}\times 86400 \text{ s}} = 30.6°$$

$u = 300$ Km/s case

$$13.2\times\frac{1.50\times10^8 \text{ km}}{300 \text{ km/s}\times 86400 \text{ s}} = 76.4°$$

(6) X rays

$$\text{time} = \frac{1.50\times10^{11}}{3.00\times10^{8}}\,\text{s} = 500\ \text{s} = 8.33\ \text{min}$$

protons

$$10\ \text{MeV} = 10\times10^{6}\times1.60\times10^{-19}\ \text{J} = 1.60\times10^{-12}\ \text{J}$$

$$\text{velocity} = \sqrt{\frac{2E}{m}} \quad \text{(non-relativistic)}$$

$$= \sqrt{\frac{2\times1.60\times10^{-12}}{1.67\times10^{-27}}}\ \text{m/s} = 4.38\times10^{7}\ \text{m/s}$$

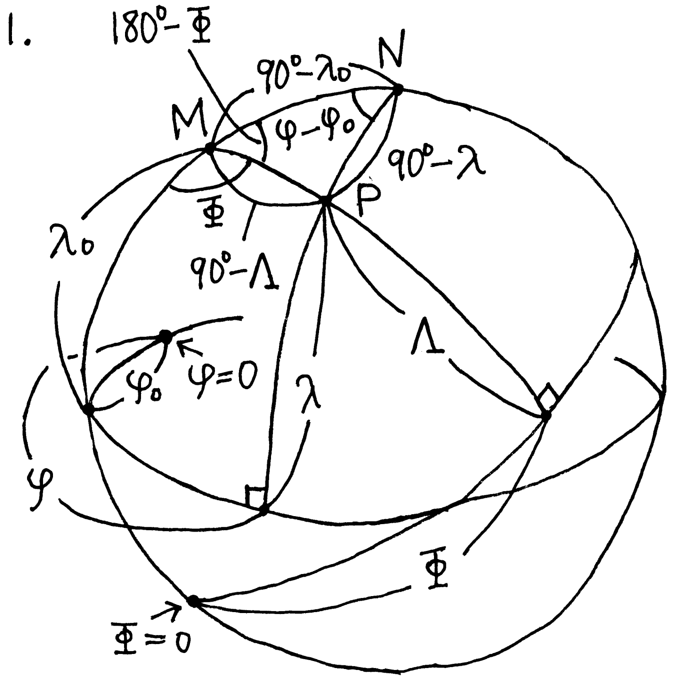
$$\text{time} = \frac{1.50\times10^{11}}{4.38\times10^{7}}\,\text{s} = 3.42\times10^{3}\ \text{s} = 57.0\ \text{min}$$

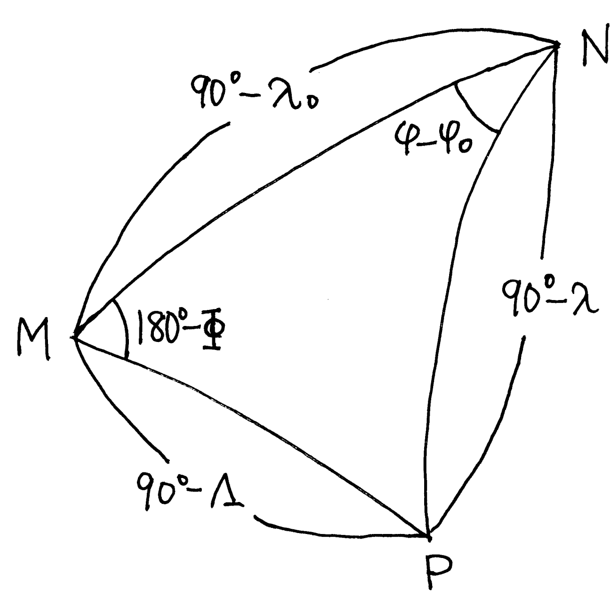
solar wind

$$\text{time} = \frac{1.50\times10^{8}\ \text{km}}{450\ \text{km/s}} = 3.33\times10^{5}\ \text{s} = 3.85\ \text{days}$$

(7)

1.





cosine law

$$\cos(90^\circ-\Lambda)=\cos(90^\circ-\lambda)\cos(90^\circ-\lambda_0)+\sin(90^\circ-\lambda)\sin(90^\circ-\lambda_0)\cos(\varphi-\varphi_0)$$

$$\therefore\ \sin\Lambda=\sin\lambda\sin\lambda_0+\cos\lambda\cos\lambda_0\cos(\varphi-\varphi_0)$$

sine law

$$\frac{\sin(90^\circ-\Lambda)}{\sin(\varphi-\varphi_0)}=\frac{\sin(90^\circ-\lambda)}{\sin(180^\circ-\Phi)}$$

$$\therefore\ \sin\Phi=\frac{\cos\lambda}{\cos\Lambda}\sin(\varphi-\varphi_0)$$

2. Bandung $\lambda=-6.9^\circ$ $\varphi=107.6^\circ$

North geomagnetic pole $\lambda_0=80.0^\circ$ $\varphi_0=-72.2^\circ$

$$\sin\Lambda=-\sin 6.9^\circ\sin 80.0^\circ+\cos 6.9^\circ\cos 80.0^\circ\cos(107.6^\circ+72.2^\circ)$$

$$=-0.2907 \qquad \Lambda=-16.9^\circ$$

(8)

1. $$B=\sqrt{B_r^2+B_\lambda^2}=B_0\left(\frac{a}{r}\right)^3\sqrt{1+3\sin^2\lambda}$$

2. $$\frac{dr}{r\,d\lambda}=\frac{B_r}{B_\lambda}=-\frac{2\sin\lambda}{\cos\lambda}$$

Let $y=\cos\lambda$. Then $\dfrac{dr}{r}=-\dfrac{2\sin\lambda\,d\lambda}{\cos\lambda}=\dfrac{2dy}{y}$

$\therefore\ r=cy^2$ $\quad c=\text{const.}$

$r=La$ at $\lambda=0^\circ$ $\quad\therefore\ c=La$

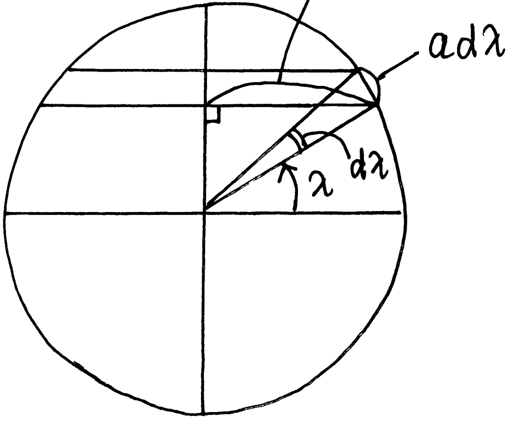
$$\therefore\ \frac{r}{a}=L\cos^2\lambda$$

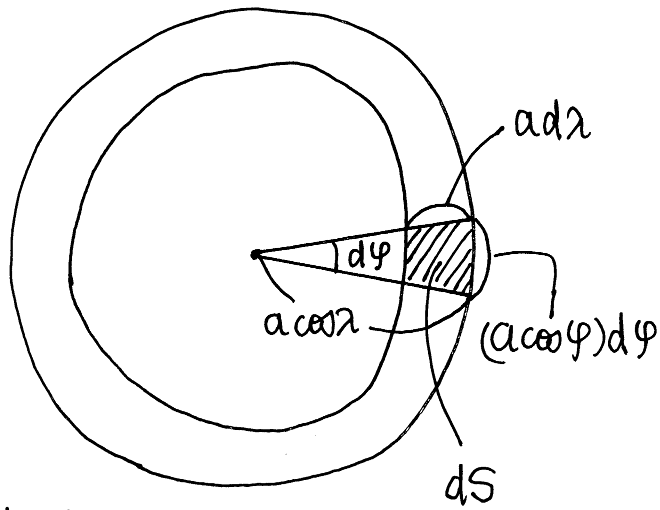
3. $\cos \Lambda = \frac{1}{\sqrt{L}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$ $\therefore \Lambda = 60°$

4. Bandung $\Lambda = -16.9°$

$$L = \frac{1}{\cos^2 \Lambda} = \frac{1}{\cos^2 16.9°} = 1.09$$

(9)





$$dS = (a\cos\lambda)d\varphi \cdot a\,d\lambda = a^2 \cos\lambda\, d\varphi\, d\lambda$$

$$\Phi_{pc} = \iint |B_r| dS = a^2 \int_0^{2\pi} d\varphi \int_\Lambda^{\frac{\pi}{2}} d\lambda \cos\lambda |B_r|$$

$$= 2\pi a^2 \int_\Lambda^{\frac{\pi}{2}} d\lambda \cos\lambda |B_r| = 2\pi a^2 \int_\Lambda^{\frac{\pi}{2}} 2B_0 \sin\lambda \cos\lambda\, d\lambda$$

$$= 2\pi a^2 B_0 \cos^2 \Lambda$$

$$\frac{\Phi_{pc}(\Lambda = 70°)}{\Phi_{pc}(\Lambda = 75°)} = \left(\frac{\cos 70°}{\cos 75°}\right)^2 = 1.74$$

The magnetic flux increases by a factor of 1.74.

(10) 1. Let $\theta \equiv 90^\circ - \lambda$.

$$W_{dip} = \int_{r \geq a} \frac{B^2}{2\mu_0} dV = \int_{r \geq a} \frac{B_0^2}{2\mu_0} \left(\frac{a}{r}\right)^6 (1+3\cos^2\theta) r^2 \sin\theta \, dr d\theta d\varphi$$

$$= \frac{B_0^2 a^6}{2\mu_0} \int_a^\infty dr \frac{1}{r^4} \int_0^\pi d\theta (1+3\cos^2\theta)\sin\theta \int_0^{2\pi} d\varphi$$

$$= \frac{B_0^2 a^6}{2\mu_0} \cdot \frac{1}{3a^3} \cdot 4 \cdot 2\pi = \frac{4\pi a^3 B_0^2}{3\mu_0}$$

$$= \frac{4\pi \times (6.38\times10^6)^3 \times (3.00\times10^{-5})^2}{3\times 4\pi \times 10^{-7}} \text{ J}$$

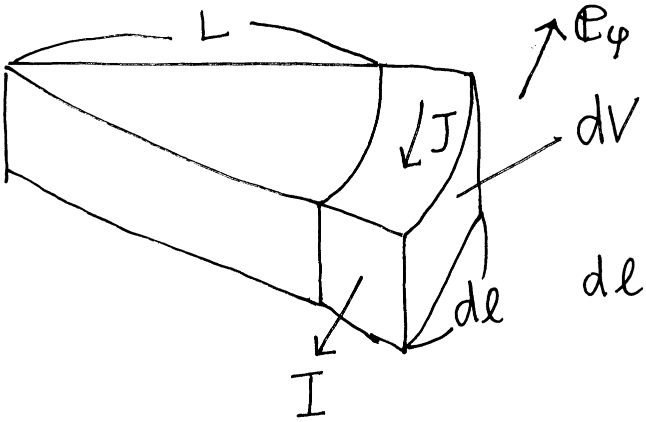
$$= 7.79 \times 10^{17} \text{ J}$$

2. current density J caused by ring current particles with a particular energy W and density n circulating on a given L-shell

$$J(L) = q n v_B = \frac{3L^2 n w}{B_0 a} \quad (-e_\varphi \text{ direction})$$

total current I

$$I(L) = J\frac{dV}{dl} \qquad \therefore \qquad I dl = J dV$$



dl: circumference element

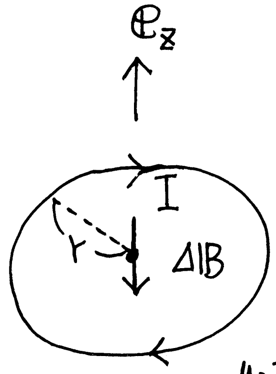
total energy W of all ring current particles at that L

$$W(L) = \int n w \, dV$$

$$\int I\,dl = \int J\,dV$$

$$I \cdot 2\pi L a = \int \frac{3L^2 n w}{B_0 a} dV = \frac{3L^2}{B_0 a} W$$

$$\therefore\ I = \frac{3LW}{2\pi B_0 a^2}$$



$$(\Delta B)_z = -\frac{\mu_0 I}{2r}$$

ΔB at the Earth's center caused by I

$$(\Delta B)_z = -\frac{\mu_0 I}{2La} = -\frac{\mu_0}{4\pi}\frac{3W}{B_0 a^3}$$

Magnetic disturbance does not explicitly depend on L.

We can replace $W(L)$ by the total energy of ring current particles W_{RC}.

$$(\Delta B)_z = -\frac{\mu_0}{4\pi}\frac{3W_{RC}}{B_0 a^3}$$

Each gyrating particle has a magnetic moment $\mu = \frac{w}{B}$.

ΔB at the Earth's center caused by μ

$$(\Delta B)_z = \frac{\mu_0}{4\pi}\frac{\mu}{L^3 a^3}$$

$$\left[\text{cf. dipole field } \mathbb{B} = \frac{\mu_0}{4\pi}\frac{M}{r^3}(-2\sin\lambda\, \mathbb{e}_r + \cos\lambda\, \mathbb{e}_\lambda)\right]$$

In the equatorial plane, $B = \frac{B_0}{L^3}$; hence $\mu = \frac{w}{B} = \frac{wL^3}{B_0}$.

$$\therefore (\Delta B)_z = \frac{\mu_0}{4\pi}\frac{1}{L^3 a^3}\frac{wL^3}{B_0} = \frac{\mu_0}{4\pi}\frac{w}{B_0 a^3}$$

For all particles at L,

$$(\Delta B)_z = \int \frac{\mu_0}{4\pi}\frac{n w}{B_0 a^3} dV = \frac{\mu_0}{4\pi}\frac{W(L)}{B_0 a^3}$$

Magnetic disturbance does not explicitly depends on L.

We can replace $W(L)$ by the total energy of ring current particles W_{RC}.

$$(\Delta \mathbb{B})_z = \frac{\mu_0}{4\pi} \frac{W_{RC}}{B_0 a^3}$$

total magnetic field disturbace at the Earth's center

$$(\Delta \mathbb{B})_z = (-3+1) \cdot \frac{\mu_0}{4\pi} \frac{W_{RC}}{B_0 a^3} = -\frac{\mu_0}{2\pi} \frac{W_{RC}}{B_0 a^3}$$

$$= -\frac{2}{3} \frac{W_{RC}}{W_{dip}}$$